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APPROXIMATE MODELS OF CMM BEHAVIOUR AND POINT CLOUD UNCERTAINTIES

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Abstract - Coordinate metrology can be thought of as a two-stage process, the first stage using a coordinate measuring system to gather coordinate data – point clouds – related to a workpiece surface, the second extracting a set of parameters or characteristics from the data. In this paper, we describe a range of straightforward, approximate models of CMM behaviour that can be used to generate variance matrices associated with point clouds. In particular, we describe models that incorporate spatial correlation to capture the smooth departure of CMM behaviour from ideal geometry. We also discuss how variance matrices associated with point clouds can be propagated through to variance matrices associated with derived features.

Keywords: coordinate metrology, spatial correlation, statistical models, uncertainty evaluation

1. INTRODUCTION

Coordinate metrology can be thought of as a two-stage process, the first stage using a coordinate measuring system to gather coordinate data $\boldsymbol{x}_i$, $i = 1,\ldots,m$, related to a workpiece surface, the second extracting a set of parameters or characteristics $\boldsymbol{a} = (a_1,\ldots,a_n)^\top$ from the data $\boldsymbol{x}_{1:m}$ using software implementing mathematical algorithms, e.g., determining the parameters associated with the best-fit cylinder to data. The evaluation of the uncertainties associated with geometric features $\boldsymbol{a}$ derived from coordinate data $\boldsymbol{x}_{1:m}$ is also a two stage process, the first in which a $3m \times 3m$ variance matrix V_X associated with the coordinate data is evaluated, the second stage in which the uncertainties associated with $\boldsymbol{x}_{1:m}$ are propagated through to those for the features $\boldsymbol{a}$ derived from $\boldsymbol{x}_{1:m}$. The uncertainties associated with the computed parameters depend on the uncertainties associated with the point coordinates as encoded in the associated variance matrix. While the true variance matrix may be difficult to evaluate, a reasonable approximation can be determined using approximate models of CMM behaviour.

2. GENERAL MODEL OF CMM MEASUREMENT

A general model of CMM measurement has the form

$$\boldsymbol{x}_i = \boldsymbol{x}_i^* + \boldsymbol{e}_i + \boldsymbol{\epsilon}_i, \quad \boldsymbol{\epsilon}_i \in \mathrm{N}(\boldsymbol{0}, \sigma_i^2 I) \qquad (1)$$

where $\boldsymbol{x}_i$ is the measured coordinates, $\boldsymbol{x}_i^*$ is the true point coordinates, $\boldsymbol{e}_i$ is a systematic effect and $\boldsymbol{\epsilon}_i$ is a random effect, $i = 1,\ldots,m$. The systematic effect $\boldsymbol{e}_i$ is taken to be approximately constant over the duration of a measurement of a part while the random effect $\boldsymbol{\epsilon}_i$ represents a sum of effects that change over a very short timescale effectively modelling the repeatability component of the CMM uncertainty.

We generalise the model in (1) to cater for the possibility that the measurements may be subject to a number of independent systematic effects that combine additively to influence the measurement result, e.g.,

$$\boldsymbol{x}_i = \boldsymbol{x}_i^* + \boldsymbol{e}_{i,B} + \boldsymbol{e}_{i,C} + \boldsymbol{e}_{i,D} + \boldsymbol{\epsilon}_i, \quad \boldsymbol{\epsilon}_i \in \mathrm{N}(\boldsymbol{0}, \sigma_i^2 I) \qquad (2)$$

We assume that the behaviour of the systematic effects can be described by a statistical model which allows use to calculate (or estimate) the contribution to the variance matrix V_X associated with $\boldsymbol{x}_{1:m}$ from the various effects. We denote by $V_{X|B}$, the variance contribution arising from $\boldsymbol{e}_{1:m,B}$, etc. For the model in (2), the variance matrix V_X can be decomposed as

$$V_X = V_{X|B} + V_{X|C} + V_{X|D} + V_{X|R}.$$

We denote by $V_{X|R}$ the diagonal variance matrix representing the variance contribution from the random effects $\boldsymbol{\epsilon}_{1:m}$.

2.1. Propagation of variances

The law of propagation of uncertainty, the basis of the GUM [1], in its multivariate setting [2] describes how uncertainties associated with the measured coordinates in (2) can be evaluated on the basis of uncertainties associated with the systematic and random effects. Suppose effects $\boldsymbol{e}_{i,B} = \boldsymbol{e}_i(\boldsymbol{b})$, $i = 1,\ldots,m$, are specified by n_B parameters $\boldsymbol{b} = (b_1,\ldots,b_{n_B})^\top$, and that a statistical model for $\boldsymbol{b}$ specifies the $n_B \times n_B$ variance matrix V_B associated with $\boldsymbol{b}$. If $G_{X|B}$ is the $3m \times n_B$ sensitivity matrix of $\boldsymbol{x}_{1:m}$ with respect to $\boldsymbol{b}$ constructed from $3 \times n_B$ matrices

$$G_{X|B,i} = \frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{b}^\top},$$

then

$$V_{X|B} = G_{X|B} V_B G_{X|B}^\top.$$

The role of the sensitivity matrix $G_{X|B}$ can be explained as follows. If the parameters $\boldsymbol{b}$ describing the systematic effects are perturbed by $\Delta\boldsymbol{b}$, then the resulting perturbation on $\boldsymbol{e}_{1:m}$ and hence $\boldsymbol{x}_{1:m}$ is given by $\Delta\boldsymbol{x}_{1:m} = G_{X|B}\Delta\boldsymbol{b}$, to first order.

Often we are interest in quantities derived from a set of point coordinates. As a consequence of the chain rule in calculus, if $\boldsymbol{a} = (a_1, \ldots, a_{n_A})^\top$ depends on $\boldsymbol{x}_{1:m}$ and $G_{A,X}$ is the $n_A \times 3m$ sensitivity matrix of $\boldsymbol{a}$ with respect to $\boldsymbol{x}_{1:m}$ then the $n_A \times n_B$ sensitivity matrix $G_{A|B}$ of A with respect to influence factors $\boldsymbol{b}$ is given by

$$G_{A|B} = G_{A|X} G_{X|B},$$

and the $n_A \times n_A$ variance matrix $V_{A|B}$ describing the variance contribution to $\boldsymbol{a}$ arising from factors $\boldsymbol{b}$ is given by

$$V_{A|B} = G_{A|B} V_B G_{A|B}^\top.$$

If the systematic effects $\boldsymbol{b}$ are perturbed by $\Delta\boldsymbol{b}$, then the derived parameters $\boldsymbol{a}$ are perturbed by $\Delta\boldsymbol{a} = G_{A|B}\Delta\boldsymbol{b}$, to first order. An important example of quantities $\boldsymbol{a}$ derived from point coordinates $\boldsymbol{x}_{1:m}$ is where $\boldsymbol{a}$ are parameters associated with a Gaussian associated feature to $\boldsymbol{x}_{1:m}$, e.g., the least squares best-fit cylinder to a data set.

3. SCALE AND SQUARENESS EFFECTS

From practical experience, it is well known that a major component of CMM behaviour relates to scale and squareness errors. The model below incorporates a global scale effect b_{aa}, axis scale effects b_{xx}, b_{yy} and b_{yz}, and three squareness effects b_{xy}, b_{xz} and b_{yz} through

$$\boldsymbol{x}_i = B(\boldsymbol{b})\boldsymbol{x}_i^* + \boldsymbol{\epsilon}_i. \tag{3}$$

with $B(\boldsymbol{b}) =$

$$\begin{bmatrix} (1 + b_{aa} + b_{xx}) & b_{xy} & b_{xz} \\ 0 & (1 + b_{aa} + b_{yy}) & b_{yz} \\ 0 & 0 & (1 + b_{aa} + b_{zz}) \end{bmatrix}, \tag{4}$$

depending on $\boldsymbol{b} = (b_{aa}, b_{xx}, b_{yy}, b_{zz}, b_{xy}, b_{xz}, b_{yz})^\top$. The $3m \times 7$ sensitivity matrix $G_{X|B}$ for this model is assembled from 3×7 matrices of the form

$$G_i = \begin{bmatrix} x_i & x_i & 0 & 0 & y_i & z_i & 0 \\ y_i & 0 & y_i & 0 & 0 & 0 & z_i \\ z_i & 0 & 0 & z_i & 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

The model is completed by specifying the variance matrix V_B associated with the scale and squareness effects. Over modest working volumes over which straightness and rotational effects are not significant, the scale and squareness model is a useful approximation. For this model, the variance contribution $V_{X|B}$ associated with a set of coordinates from $\boldsymbol{b}$ is given by

$$V_{X|B} = G_{X|B} V_B G_{X|B}^\top, \tag{6}$$

where $G_{X|B}$ is the $3m \times 7$ sensitivity matrix constructed from G_i defined as in (5). In practice, we usually avoid forming this matrix explicitly but perform calculations using $G_{X|B}$ and V_B.

4. PROBE QUALIFICATION EFFECTS

For error models with an explicit dependence on the probe offset $\boldsymbol{p}_k$, the fact that the probe configuration geometry is usually determined in probe qualification experiments means that there will be uncertainties associated with estimates of the offsets. If $\boldsymbol{x}_i$ is a measurement using the kth probe, then the uncertainty contribution arising from the probe qualification can be modelled as

$$\boldsymbol{x}_i = \boldsymbol{x}_i^* + \boldsymbol{p}_k + \boldsymbol{e}_{PQ,k} + \boldsymbol{\epsilon}_i, \quad \boldsymbol{e}_{PQ,k} \in \mathrm{N}(\boldsymbol{0}, \sigma_{PQ,k}^2 I), \tag{7}$$

where $\boldsymbol{p}_k$ is the calibrated probe offset vector for the kth probe and $\boldsymbol{e}_{PQ,k}$ models the (unknown) difference between the actual probe offset and its calibrated value, $k = 1, \ldots, n_P$. An important feature of the model is that all measurements with the kth probe are associated with the same systematic effect $\boldsymbol{e}_{PQ,k}$. The variance contribution associated with probe qualification is given by

$$V_{X|PQ} = G_{X|PQ} V_{PQ} G_{X|PQ}^\top$$

where $V_{X|PQ}$ is the $3n_P \times 3n_P$ variance matrix associated with the systematic effects $\boldsymbol{e}_{PQ,k}$ and $G_{X|PQ}$ is the $3m \times 3n_P$ sensitivity matrix. The variance matrix V_{PQ} is a diagonal matrix with the 3×3 matrix $\sigma_{PQ,k}^2 I$ in the kth diagonal block. If the ith measurement is associated with the kth probe, then

$$G_{X|PQ}(3i - 2 : 3i, 3k - 2 : 3k) = I$$

the 3×3 identity matrix, and all other elements in these three rows are zero.

5. SPATIAL CORRELATION MODELS

Gaussian process models [3, 4] can be used to develop empirical models of behaviour that incorporate spatial or temporal correlation. If an effect e is associated with a spatial location $\boldsymbol{x}$, then the correlation between effects e and e' is evaluated as

$$\mathrm{corr(e, e')} = k(\boldsymbol{x}, \boldsymbol{x}'|\boldsymbol{\sigma})$$

where k is a correlation kernel depending on statistical parameters $\boldsymbol{\sigma}$. Often k depends on $\boldsymbol{x}$ and $\boldsymbol{x}'$ through $\|\boldsymbol{x} - \boldsymbol{x}'\|$, e.g.

$$\mathrm{cov}(e, e') = k(\boldsymbol{x}, \boldsymbol{x}') = \sigma_1^2 \exp\{-\|\boldsymbol{x} - \boldsymbol{x}'\|^2/\sigma_2^2\}. \tag{8}$$

The strength of the correlation between e and e' depends on the distance between $\boldsymbol{x}$ and $\boldsymbol{x}'$: the closer $\boldsymbol{x}$ is to $\boldsymbol{x}'$, relative to σ_2, the stronger the correlation between e and e'.

A GP model can be used to supplement a parametric model $\boldsymbol{e}(\boldsymbol{b})$ for the systematic effects, e.g., a scale and squareness error model considered in section 3, in which the role of the GP model is to simulate behaviour not captured by the parametric model.

5.1. GP models for location errors

We can apply a GP model for CMM behaviour as follows with

$$\boldsymbol{x}_i = \boldsymbol{x}_i^* + \boldsymbol{e}_i + \boldsymbol{\epsilon}_i, \tag{9}$$

where the systematic effects are spatially (and sometimes temporally) correlated. In general, the covariance applies only along the same axis with the x-, y- and z-coordinates of $\boldsymbol{e}$ mutually independent. The covariance with e_x with e_x' could be modelled as

$$\mathrm{cov}(e_x, e_x') = k(\boldsymbol{x}, \boldsymbol{x}'|\boldsymbol{\sigma}_x) = \sigma_{1,x}^2 \exp\left\{-\|\boldsymbol{x}-\boldsymbol{x}'\|^2/\sigma_{2,x}^2\right\},$$

for example, where $\sigma_{2,x}$ defines the length scale for the correlation in the x-coordinate. Note that in this model, the strength of the correlation in the effects e_x depends on the distance $\|\boldsymbol{x}-\boldsymbol{x}'|$ in three dimensions, not the distance along the x-axis.

Let D be the $m \times m$ matrix of distances with

$$D_{ij} = \|\boldsymbol{x}_i - \boldsymbol{x}_j\|.$$

The variance contribution V_{XT} from $\boldsymbol{e}_{1:m}$ to the x-coordinates of $\boldsymbol{x}_{i:m}$ is given by

$$V_{XT,x} = \sigma_{1,x}^2 \exp\left\{-D^2/\sigma_{2,x}^2\right\}$$

where the calculations associated with D are made element-wise. The contribution to the y- and z-components are of exactly the same form. The matrix V_{XT} is assembled from $V_{XT,x}$ $V_{XT,y}$ and $V_{XT,z}$, with all other elements zero since we assume that the systematic effects associated with the x-coordinates are independent from those associated with the y- and z-coordinates, etc.

5.2. Gaussian process model for location and rotation errors

The GP models in section 5.1 used, perhaps, with a simple parametric error model can simulate a wide range of plausible CMM behaviour but it relates only to one probing configuration and does not, without modification, allow us to evaluate the uncertainties associated with different probe configurations. An extension of the model is to use GP models for both the location and rotation errors:

$$\boldsymbol{x}_i = \boldsymbol{x}_i^* + \boldsymbol{e}_i + R(\boldsymbol{\alpha}_i)\boldsymbol{p} + \boldsymbol{\epsilon}_i, \tag{10}$$

where $\boldsymbol{\alpha}_i = (\alpha_{i,x}, \alpha_{i,y}, \alpha_{i,z})^\top$ represents three spatially correlated rotation errors acting on the probe offset vector $\boldsymbol{p}$ through the rotation matrix

$$R(\boldsymbol{\alpha}_i) = R_z(\alpha_{i,z})R_y(\alpha_{i,y})R_x(\alpha_{i,x}), \tag{11}$$

the product of rotations about each of the three coordinate axes.

We note that if the variance matrix associated with $\boldsymbol{\alpha} = (\alpha_x, \alpha_y, \alpha_z)^\top$ with $\boldsymbol{\alpha} = \boldsymbol{0}$ is $V_{\boldsymbol{\alpha}}$, then the variance matrix $V_{\boldsymbol{p}}$ associated with $R(\boldsymbol{\alpha})\boldsymbol{p}$, with $R(\boldsymbol{\alpha})$ as in (11), is given by $GV_{\boldsymbol{\alpha}}G^\top$ where

$$G = \begin{bmatrix} 0 & p_z & -p_y \\ -p_z & 0 & p_x \\ p_y & -p_x & 0 \end{bmatrix}, \quad \boldsymbol{p} = (p_x, p_y, p_z)^\top. \tag{12}$$

The explicit dependence on the probe offset allows different probe configurations to be modelled. For this case, it is important to note that the spatial correlation is dependent on $\|\boldsymbol{x}_i^* - \boldsymbol{x}_q^*\|$, not $\|\boldsymbol{x}_i - \boldsymbol{x}_q\|$, following (10). For different probe configurations we have

$$\|\boldsymbol{x}_i^* - \boldsymbol{x}_q^*\| \doteq \|(\boldsymbol{x}_i - \boldsymbol{p}_{k(i)}) - (\boldsymbol{x}_q - \boldsymbol{p}_{k(q)})\|,$$

where $\boldsymbol{p}_{k(i)}$ denotes the probe configuration associated with the ith measurement, etc.

If $\tilde{V}_{XR}$ is the $3m \times 3m$ variance matrix associated with $\boldsymbol{\alpha}_{1:m}$ determined from the correlation kernel (or otherwise), then the variance contribution to the measurements $\boldsymbol{x}_{1:m}$ is given by

$$V_{XR} = G_{XR}\tilde{V}_{XR}G_{XR}^\top,$$

where G_{XR} is a $3m \times 3m$ block-diagonal matrix. If the ith measurement is associated with the kth probe, then the 3×3 ith diagonal is equal to G_k, where G_k is constructed from $\boldsymbol{p}_k$ as in (12).

5.3. Gaussian process model for probing effects

The operation of the probe system is likely also to make a variance contribution. While the CMM geometric errors are likely to vary smoothly with location, the probing errors are likely to vary smoothly with the probing direction, usually designed to be normal to the surface being probed. We can augment the model in (9) to one of the form

$$\boldsymbol{x}_i = \boldsymbol{x}_i^* + \boldsymbol{e}_i + e_{P,i}\boldsymbol{n}_i + \boldsymbol{\epsilon}_i, \tag{13}$$

where $e_{P,i}$ is a systematic effect associated with probing and $\boldsymbol{n}_i$ is the unit normal probing direction. The correlation between effects $e_{P,i}$ and $e_{P,j}$ depends the spatial separation $\|\boldsymbol{n}_i - \boldsymbol{n}_j\|$ if both measurements use the same probe. We assume that probing effects associated with different probes are statistically independent (although there may be situations where some statistical dependence would be expected). The variance matrix associated with spatially correlated probing effects is denoted by V_{XP}.

6. COMBINED EFFECTS

We can write the variance matrix V_X incorporating all the effects considered above as

$$\begin{aligned} V_X =& V_{XT} + V_{XR} + V_{XP} + \ldots \\ & G_{X|B}V_BG_{X|B}^\top + G_{X|PQ}V_{PQ}G_{X|PQ}^\top + V_R, \end{aligned}$$

where the first three variance matrices or the right are derived from spatially correlated location, rotation and probing effects, and the second three are the contributions from scale and squareness effects, probe qualification effects and independent random effects, respectively. For some cases, not all effects need to be considered. For example, for measurements using a single probe, rotational effects and probe qualifications need to be calculated. While the model does have some degree of complexity, all the variance matrices can be calculated using direct calculations based

	R	S	G
$u(x_0)/0.001$ mm	1.8	0.2	3.0
$u(y_0)/0.001$ mm	1.8	0.2	3.5
$u(r_0)/0.001$ mm	1.3	2.5	1.7

Table 1. Uncertainty calculations associated with a cylinder parameters for three uncertainty models.

on, for example, the point coordinates, the distances between points, etc.

If $G_{A|X}$ us the sensitivity matrix associated with a feature vector $\boldsymbol{a}$ with respect to coordinates $\boldsymbol{x}_{1:m}$, then the variance matrix V_A associated with $\boldsymbol{a}$ can also be decomposed as

$$\begin{aligned} V_A =& V_{A|XT} + V_{A|XR} + V_{A|XP} + ... \\ & G_{A|B} V_B G_{A|B}^\top + G_{A|PQ} V_{PQ} G_{A|PQ}^\top + G_{A|X} V_R G_{A|X}^\top, \end{aligned}$$

where $V_{A|XT} = G_{A|X} V_{XT} G_{A|X}^\top$, etc., and $G_{A|B} = G_{A|X} G_{X|B}$, etc. Thus $G_{A|PQ} V_{PQ} G_{A|PQ}^\top$ is the variance contribution to V_A arising from probe qualification effects, for example.

7. FEATURES DERIVED FROM COORDINATE DATA

If parameters $\boldsymbol{a}$ are determined from fitting a geometric element to data according to the least squares criterion minimising

$$\sum_i d_i^2(\boldsymbol{x}_i, \boldsymbol{a}), \quad d_i(\boldsymbol{x}_i, \boldsymbol{a}) = \boldsymbol{n}_i^\top (\boldsymbol{x}_i - \boldsymbol{s}(\boldsymbol{u}_i^*, \boldsymbol{a}))$$

then if J is the Jacobian matrix at the solution with $J_{ij} = \partial d_i / \partial a_j$ and N the projection matrix with $N_{i,3i-2:3i} = \boldsymbol{n}_i^\top$ then the sensitivity matrix $G_{A|x}$ of $\boldsymbol{a}$ with respect to $\boldsymbol{x}_{1:m}$ is given by $G_{A|X} = (J^\top J)^{-1} J^\top N$.

7.1. Cylinder features

A cylinder [5] can be specified by five parameters $\boldsymbol{a} = (x_0, y_0, \alpha, \beta, r_0)^\top$ where (x_0, y_0) are coordinates where the cylinder axis cuts the plane $z = 0$, α and β are rotation angles about the x- and y-axes respectively, defining the axis direction and r_0 its radius. We have constructed three CMM uncertainty models: random effects only, (R), scale and squareness effects (S), and spatially correlated location effects (G). All three models are such that the uncertainty associated with each coordinate = 0.004 mm, approximately. The uncertainties associated with x_0, y_0 and r_0 for the three models are given in Tab. 1. While the uncertainties associated with each coordinate are the same, approximately, for each model, the effect on the cylinder parameter estimates are significantly different, due to the different types of correlation associated with the point cloud uncertainties.

8. CONCLUSIONS

In this paper, we have described a range of straightforward, approximate models of CMM behaviour that can be used to generate variance matrices associated with point clouds. In particular, we have described models that incorporate spatial correlation to capture the smooth departure of CMM behaviour from ideal geometry. We have also discussed how variance matrices associated with point clouds can be propagated through to variance matrices associated with derived features. Through the collaborative EUCoM project (see acknowledgements below), work is continuing to validate the models from a comprehensive set measurement experiments involving a range of calibrated artefacts. By providing plausible prior models of CMM behaviour, it is hoped to be able to evaluate uncertainties associated with derived features that are more realistic and help practitioners determine if a particular instrument or measurement strategy is fit for purpose as well as supporting the traceability of coordinate measurement to the metre.

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